LAMINAR BOUNDARY LAYER AT A PERMEABLE SURFACE

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A universal equation for the boundary layer with continuous suction is proposed and solved (to the first approximation) in Crocco variables.

A new approach to solving problems of a boundary layer at a permeable surface [1] is based on the parametric method by L. G. Loitsyanskii [2] and its gist is to reduce the fundamental equations to a form which depends neither on the specific velocity distribution in the main stream nor on the velocity of the fluid across the surface. The effect of the main stream and of the injection or suction rate is accounted for by two series of parameters. A universal equation was used in [1] for generalizing the flow function in ordinary similarity variables. No numerical solution was obtained there, however; the author went only as far as expanding the solution into power series.

Here we will derive a universal equation in Crocco variables for a laminar isothermal boundary layer at a permeable surface and will show numerical results obtained with the aid of a digital computer.

We define injection to or suction from the boundary layer by specifying the transverse velocity $v_0(x)$ of fluid flow through the surface. Formulation of the problem in Crocco variables for a permeable wall differs from such a formulation for an impermeable wall by the presence of velocity $v_0(x)$ in the boundary condition at the wall.

The original equation is

$$u\frac{\partial}{\partial x}\left(\frac{1}{\tau}\right)+\frac{1}{\mu\rho}\cdot\frac{\partial^2\tau}{\partial u^2}=-UU'\frac{\partial}{\partial u}\left(\frac{1}{\tau}\right), \tag{1}$$

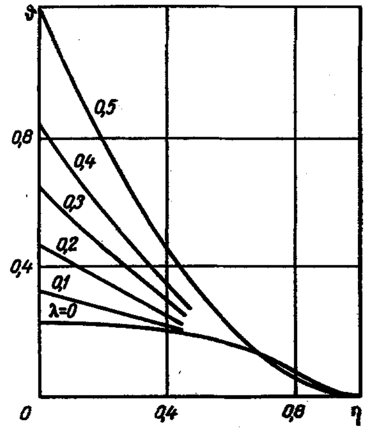

Fig. 1

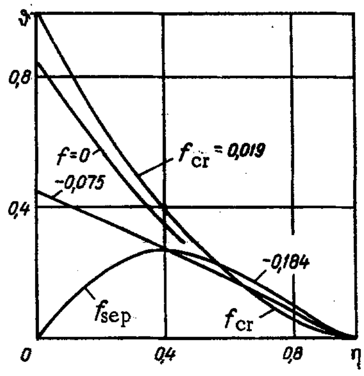

Fig. 2

Fig. 1. Distribution of frictional stress ϑ across a boundary layer at a plate when suction occurs.

Fig. 2. Distributions of frictional stress in a boundary layer for $\lambda = 0.4$; f_{cr} and f_{sep} are values of the form factor f at the stagnation point and at the separation point respectively.

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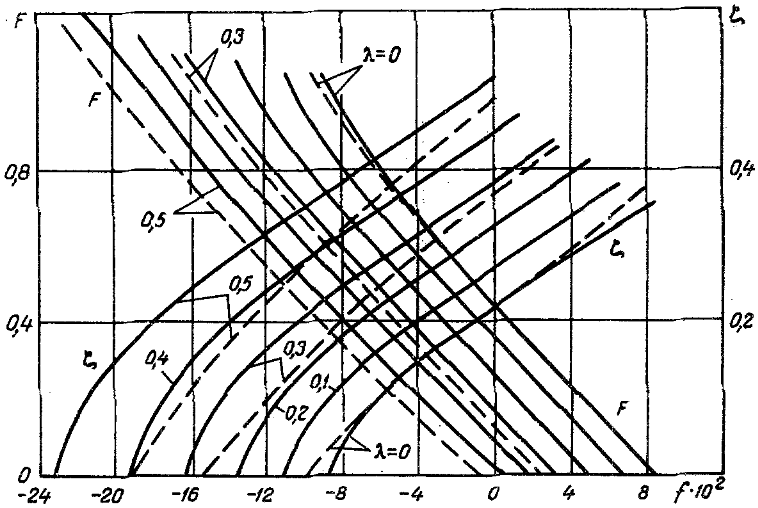

Fig. 3. Fundamental characteristics of a boundary layer: F(f, λ) and ζ(f, λ). Dashed curves represent data in [1].

with

$$\tau \frac{\partial \tau}{\partial u} = -\rho\mu UU' + \rho v_0 \tau \quad \text{at} \quad u = 0, \quad \tau = 0 \quad \text{at} \quad u = U.$$

We introduce into Eq. (1) the dimensionless variables

$$\eta = \frac{u}{U}, \qquad \omega = \frac{\tau h}{U\mu}.$$

A transformation by means of

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x}\right) - \frac{U'}{U}\eta\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial u} = \frac{1}{U}\cdot\frac{\partial}{\partial \eta},$$

$$\frac{\partial^2}{\partial u^2} = \frac{1}{U^2}\cdot\frac{\partial^2}{\partial \eta^2},$$

yields instead of (1):

$$\omega^2\frac{\partial^2\omega}{\partial\eta^2} - (1-\eta^2)\frac{U'h^2}{\nu}\cdot\frac{\partial\omega}{\partial\eta} + \frac{U'h^2}{\nu}\left(\frac{h'U}{hU'} - 1\right)\eta\omega = \eta\frac{Uh^2}{\nu}\cdot\frac{\partial\omega}{\partial x},$$

$$\omega\frac{\partial\omega}{\partial\eta} = -\frac{U'h^2}{\nu} + \frac{v_0 h}{\nu}\omega \quad \text{at} \quad \eta = 0, \tag{2}$$

$$\omega = 0 \quad \text{at} \quad \eta = 1 \quad \left(\nu = \frac{\mu}{\rho}\right).$$

We will now show that Eq. (2) with the respective boundary conditions can be reduced to a universal form which does not depend on the specific velocity distributions U(x) and v_0(x).

First of all, we use the momentum equation, which will be derived directly from (1) by twice integrating both parts with respect to u.

With the boundary conditions in (1) we have

$$\int_0^U\left(\int_0^u \frac{\partial}{\partial x}\left(\frac{1}{\tau}\right)du\right)du + U\int_0^U\frac{du}{\tau} = \frac{1}{\mu}\left(\frac{\tau_0}{\rho} + Uv_0\right) \tag{3}$$

(τ_0 denotes the value of τ at the wall). The first integral in (3) is evaluated by parts

$$\int_0^U\left(\int_0^u \frac{\partial}{\partial x}\left(\frac{1}{\tau}\right)du\right)du = U\int_0^U u\frac{\partial}{\partial x}\left(\frac{1}{\tau}\right)du - \int_0^U u^2\frac{\partial}{\partial x}\left(\frac{1}{\tau}\right)du. \tag{4}$$

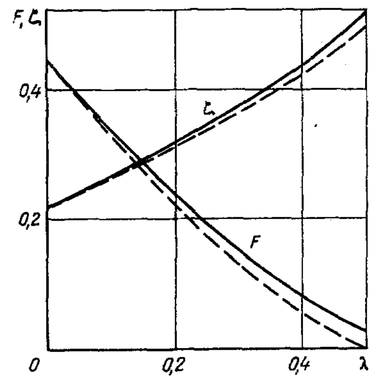

Fig. 4. Comparison of approximate with exact $F(\lambda)$ and $\zeta(\lambda)$ curves for a plate. Dashed curves represent data in [4].

Changing the order of differentiations and integrations on the left-hand side of (4), together with a few additional operations, will reduce (3) to

$$\frac{d}{dx}\int_0^U u(U-u)\frac{\mu}{\tau}\,du + U'\int_0^U (U-u)\frac{\mu}{\tau}\,du = \frac{\tau_0}{\rho} + Uv_0.$$

Introducing the conventional thicknesses of the boundary layer, namely:

the displacement thickness

$$\delta_c^* = \int_0^U \left(1 - \frac{u}{U}\right)\frac{\mu}{\tau}\,du$$

the momentum thickness

$$\delta_c^{**} = \int_0^U \frac{u}{U}\left(1 - \frac{u}{U}\right)\frac{\mu}{\tau}\,du,$$

we arrive at the following momentum equation in Crocco variables:

$$U\frac{d\delta_c^{**}}{dx} + U'\delta_c^{**}(2+H) - v_0 = \frac{\tau_0}{\rho U}, \quad H = \frac{\delta_c^*}{\delta_c^{**}}, \tag{5}$$

which could also be derived from the integral momentum condition in ordinary variables.

We will now define the quantity h:

$$h = \frac{\delta_c^{**}}{B}, \tag{6}$$

and examine the generalizing-similarity representation of the quantity ω as a function of the similarity variables:

$$\omega = \omega(\eta;\, f_1,\, f_2,\, \ldots,\, \lambda_1,\, \lambda_2,\, \ldots). \tag{7}$$

The following quantities serve here as the similarity parameters:

$$f_k = U^{k-1}\left(\frac{d^kU}{dx^k}\right)\left(\frac{\delta_c^{**2}}{\nu}\right)^k, \tag{8}$$

$$\lambda_k = -U^{k-1}\frac{d^{k-1}}{dx^{k-1}}\left(\frac{v_0}{\sqrt{\nu}}\right)\left(\frac{\delta_c^{**2}}{\nu}\right)^{k-\frac{1}{2}} \quad (k = 1,\, 2,\, \ldots). \tag{9}$$

The f_k-parameters are related to the velocity distribution in the main stream, while the λ_k-parameters are related to the distribution of surface injection ($\lambda_1 < 0$) or surface suction ($\lambda_1 > 0$). Functions $U(x)$ and $v_0(x)$ are assumed analytic.

With the aid of (7)-(9), the momentum equation can be represented in any one of the following forms:

$$\frac{\delta_c^{**\prime}}{\delta_c^{**}} = \frac{U'F}{2Uf_1}, \tag{10}$$

$$z^{**\prime} = \frac{F}{U}, \quad z^{**} = \frac{\delta_c^{**2}}{\nu}, \tag{11}$$

where

$$F = 2[\zeta - (2+H)f_1 - \lambda_1], \quad \zeta = B\omega(0;\, f_1,\, f_2,\, \ldots,\, \lambda_1,\, \lambda_2,\, \ldots),$$

$$H = \frac{\delta_c^*}{\delta_c^{**}} = \frac{1}{B}\int_0^U \left(1 - \frac{u}{U}\right)\frac{\mu U}{\delta_c^{**}\tau}\,d\left(\frac{u}{U}\right) = \frac{1}{B}\int_0^1 \frac{1-\eta}{\omega}\,d\eta,$$

$$\eta = \frac{u}{U}, \quad f_1 = \frac{U'\delta_c^{**2}}{\nu}, \quad \lambda_1 = -\frac{v_0\delta_c^{**}}{\nu}.$$

Parameters f_k and λ_k satisfy the recurrence relations

$$\frac{U}{U'} f_1 f_k' = [(k-1) f_1 + kF] f_k + f_{k+1} = \theta_k,$$
$$\frac{U}{U'} f_1 \lambda_k' = \{(k-1) f_1 + [(2k-1)/2] F\} \lambda_k + \lambda_{k+1} = \chi_k, \qquad (12)$$

which follow from (8) and (9) if the momentum equation in form (11) is used.

We now insert the generalizing-similarity representation (7) into Eq. (1). Replacing here the derivatives with respect to x by derivatives with respect to f_k and λ_k in accordance with the relation

$$\frac{\partial}{\partial x} = \sum_{k=1}^{\infty} \left(f_k' \frac{\partial}{\partial f_k} + \lambda_k' \frac{\partial}{\partial \lambda_k} \right)$$

and taking into consideration (6), (10), (12), we arrive at the desired universal equation

$$B^2 \omega^2 \frac{\partial^2 \omega}{\partial \eta^2} - (1-\eta^2) f_1 \frac{\partial \omega}{\partial \eta} + \frac{F - 2f_1}{2} \eta \omega = \eta \sum_{k=1}^{\infty} \left(\theta_k \frac{\partial \omega}{\partial f_k} + \chi_k \frac{\partial \omega}{\partial \lambda_k} \right),$$
$$\omega \frac{\partial \omega}{\partial \eta} = -\frac{f_1}{B^2} - \frac{\lambda_1}{B} \omega \quad \text{at} \quad \eta = 0, \quad \omega = 0 \quad \text{at} \quad \eta = 1.$$

For a more convenient numerical integration, we make another change of variables in this equation by letting

$$\omega^2 = \vartheta.$$

As a result, we finally obtain for a laminar boundary layer at a permeable surface the following universal equation in Crocco variables:

$$B^2 \vartheta \frac{\partial^2 \vartheta}{\partial \eta^2} - \frac{B^2}{2} \left(\frac{\partial \vartheta}{\partial \eta} \right)^2 - (1-\eta^2) f_1 \frac{\partial \vartheta}{\partial \eta} + (F - 2f_1) \eta \vartheta = \eta \sum_{k=1}^{\infty} \left(\theta_k \frac{\partial \vartheta}{\partial f_k} + \chi_k \frac{\partial \vartheta}{\partial \lambda_k} \right),$$
$$\frac{\partial \vartheta}{\partial \eta} = -\frac{2f_1}{B^2} - \frac{2\lambda_1}{B} \sqrt{\vartheta} \quad \text{at} \quad \eta = 0, \quad \vartheta = 0 \quad \text{at} \quad \eta = 1,$$
$$\vartheta = \vartheta^{(0)}(\eta) \quad \text{at} \quad f_1 = f_2 = \ldots = 0, \quad \lambda_1 = \lambda_2 = \ldots = 0, \qquad (13)$$

where $\vartheta^{(0)}(\eta)$ may be treated as the solution to the ordinary differential equation of a longitudinal stream around a plate

$$\vartheta^{(0)} \frac{d^2 \vartheta^{(0)}}{d\eta^2} - \frac{1}{2} \left(\frac{d\vartheta^{(0)}}{d\eta} \right) + 2\eta \vartheta^{(0)} = 0,$$
$$\frac{d\vartheta^{(0)}}{d\eta} = 0 \quad \text{at} \quad \eta = 0, \quad \vartheta^{(0)} = 0 \quad \text{at} \quad \eta = 1. \qquad (14)$$

This equation is obtained from (13) with $f_k = \lambda_k = 0$ ($k = 1, 2, \ldots$), if the magnitude of the constant B is chosen on the basis of the condition

$$F(0;\ 0,\ \ldots,\ 0,\ \ldots) = 2B^2 = 2B\sqrt{\vartheta^{(0)}(0)}, \quad B = 0.470.$$

The characteristic functions F, ζ, and H are defined by the solution to Eq. (13) according to

$$F = 2[\zeta - (2 + H) f_1 - \lambda_1], \qquad \zeta = B\sqrt{\vartheta(0;\ f_1,\ f_2,\ \ldots,\ \lambda_1,\ \lambda_2,\ \ldots)},$$
$$H = \frac{1}{B} \int_0^1 \frac{1-\eta}{\sqrt{\vartheta}} d\eta. \qquad (15)$$

Equation (13) is simpler than the universal equation in ordinary variables [1]. In terms of Crocco variables it is one order lower.

For $f_k \neq 0$, $\lambda_k = 0$ ($k = 1, 2, \ldots$) Eq. (13) becomes the Loitsyanskii equation [2].

Equation (13) was integrated with the aid of a BÉSM-2M computer, in an approximation including only the parameters f_1, λ_1 and considering only the local effect of suction λ_1 as the latter was varied from 0 to 0.5.

We note that Eq. (13) has a singularity at point $\eta = 1$, since the coefficient of the first derivative becomes zero then. Calculations have shown that this does not cause any particular difficulties in the numerical integration and it affects the process only when $\lambda_1 = 0$.

For an approximation of the differential equation

$$B^2\vartheta \frac{\partial^2\vartheta}{\partial\eta^2} - \frac{B^2}{2}\left(\frac{\partial\vartheta}{\partial\eta}\right)^2 - (1-\eta^2) f \frac{\partial\vartheta}{\partial\eta} + (F - 2f)\eta\vartheta = \eta F f \frac{\partial\vartheta}{\partial f},$$

$$\frac{\partial\vartheta}{\partial\eta} = -\frac{2f}{B^2} - \frac{2\lambda}{B}\sqrt{\vartheta} \quad \text{at} \quad \eta = 0, \quad \vartheta = 0 \quad \text{at} \quad \eta = 1, \tag{16}$$

$$\vartheta = \vartheta^{(0)}(\eta) \quad \text{at} \quad f = 0, \quad \lambda = 0$$

(the subscripts to parameters f and λ have, of course, been omitted) which corresponds to the proposed approximation, we used the three-point scheme. The resultant system of equations was solved by the elimination method with iterations on each layer. First we calculated the zero layer, i.e., we solved the problem of a longitudinal stream around a plate, with a parabola taken as the initial approximation. The transverse direction was covered in 0.005 steps. Curves of $\vartheta(\eta)$ for a plate are shown in Fig. 1 with various values of λ. The longitudinal problem was solved in $\Delta f = 0.005$ steps from point $f = 0$ to the right to the point where $F = 0$ and to the left to the separation point. The steps were narrowed upon approaching the stagnation point and the separation point. The trend of function $\vartheta(\eta)$ at several values of f is shown in Fig. 2 for $\lambda = 0.4$.

Solving the specific problem by the parametric method reduces to an integration of the ordinary first-order differential equation

$$\frac{dz^{**}}{dx} = \frac{F}{U}, \tag{17}$$

$$z^{**} = z^{**(0)} \quad \text{at} \quad x = x^{(0)}.$$

Curves of $F(f, \lambda)$ and $\zeta(f, \lambda)$ have been calculated by formulas (15) and are shown in Fig. 3. The dashed curves represent data in [1], verified on an example of a boundary layer subject to suction at a cylindrical surface with a sinusoidally varying velocity in the main stream.

The cylinder problem in an exact formulation has been solved by R. M. Terrill [3] for $v_0 = 0.5$ and $v_0 = 0$. The maximum value of the suction parameter λ at the separation point is 0.318 at $v_0 = 0.5$. A comparison of the solution by the proposed method as well as the solution by the method in [1] with the exact solution is limited to only this value of parameter λ.

For a suction rate $v_0 = 0.5$, the proposed method yields a distribution of frictional stress at the cylinder surface which is almost identical to the distribution obtained by the exact method, and this includes the region near the separation point. At $v_0 = 0$ ($\lambda = 0$) the frictional stress within the diffusion zone of the boundary layer is lower and separation is forestalled ($x_S = 1.79$ instead of the exact $x_S = 1.82$).

The method in [1], on the contrary, yields better results in determining the separation point at low suction rates ($x_S = 1.81$ at $v_0 = 0$); the error of the method increases with larger value of λ and leads to premature separation.

For the case of a permeable plate, our results can be compared with the exact solution in [4] over the entire range of λ values (Fig. 4). The proposed method yields somewhat higher values of $F(\lambda)$ and $\zeta(\lambda)$. The maximum error does not exceed 3.5%.

NOTATION

$\tau = u(\partial u/\partial y)$	is the shearing stress in the boundary layer;
x, y	are the longitudinal and transverse coordinate respectively;
u	is the longitudinal velocity component;
v_0	is the injection or suction rate;
U	is the velocity of main stream;
η	is the dimensionless velocity;
ω	is the generalizing-similarity representation of friction stress;
ρ	is the density;

μ, ν	are the dynamic and kinematic viscosity respectively;
h	is the conventional thickness of boundary layer;
δ_c^*	is the displacement thickness, in Crocco variables;
δ_c^{**}	is the momentum thickness;
$z^{**} = \delta_c^{**}/\nu$;	
F, ζ, H	are the boundary-layer characteristics;
f_k, λ_k	are the parameters;
B	is the normalizing constant;
$\vartheta = \omega^2$;	
θ_k, χ_k	are the right-hand sides of recurrence relations.

Subscripts

0	refers to the wall;
(0)	refers to the initial value;
k	consecutive number;
x_s	coordinate of the separation point;
'	derivative with respect to x.

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